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Guest Editor's Note
Clifford Algebras and Their Applications[1]

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The history and immediate future of the International Conferences on Clifford Algebras and Their Applications. Seven topical sessions in Ixtapa. Dirac operator: cross relations. Polemic guide: signature change, quasigroups, pseudotwistors. Clifford cogebra, coconnection and Dirac operator for Clifford cogebra.

1. HISTORICAL REMARKS

This journal issue has grown out of the Fifth International Conference on Clifford Algebras and Their Applications held in Ixtapa, México, in summer 1999. The history (and immediate future) of these conferences is as follows:

1985 The First Conference was organized by J. S. Roy Chisholm at the University of Kent, Canterbury, United Kingdom.

1989 The Second Conference was organized by Artibano Micali at the Université des Sciences et Techniques du Languedoc, Montpellier, France.

1993 The Third Conference was organized by Richard Delanghe at the Universiteit Gent, Deinze, Belgium.

1996 The Fourth Conference was organized by Klaus Habetha at the

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1999 The Fifth Conference was organized by Zbigniew Oziewicz in Ixtapa, Mexico.

2002 The Sixth Conference is organized by Rafał Abłamowicz (jointly with John Ryan), at the Tennessee Technological University, Cookeville, Tennessee; see http://math.tntech.edu/rafal/mexico/mexico.html.

A poster announcing the Fifth Conference on Clifford Algebras in 1999 appeared in 1997 with an International Advisory Board the same as the Editorial Board of the journal *Advances in Applied Clifford Algebra*, edited by Jaime Keller in México since 1991. However, in February 1999, 3 months before the Conference date, the organization fell into a crisis, with no chairman, no expected funds, and no local organizing committee in México. The Conference was almost canceled. That it was not is due to the efforts of a scientific committee in Mexico consisting of **Marcelo A. Aguilar, Raymundo Bautista, Zbigniew Oziewicz, José Antonio de la Peña Mena, Marcos Rosenbaum, Enrique Ramirez de Arellano, Adolfo Sanchez Valenzuela, Eduardo Bayro-Corrochano, and Garret Sobczyk**.

The Ixtapa Conference was supported by:

1. The Departamento de Matematicas del Centro de Investigación y de Estudios Avanzados del Instituto Politecnico Nacional, Mexico City. In April 1999, a 1-year student strike began at the Universidad Nacional Autonoma de México; without the generous help of **Enrique Ramirez de Arellano**, who offered an office and secretarial help in his Departamento de Matematicas, the organization of the Conference would not have been possible.
2. The Instituto de Mátematicas de Universidad Nacional Autonoma de México, Mexico City and personally by **Marcelo A. Aguilar** and **José Antonio de la Peña Mena**.
3. The Centro de Investigación en Matemáticas, Guanajuato, particularly by **Eduardo Bayro Corrochano** and **Adolfo Sanchez Valenzuela**.

The participants would like to thank most cordially **A. Irma Vigil de Aragón** and student-volunteers Rosalia Flores Ballesteros, Elizabeth Rivas Martinez, Claudia Rosas, and Ricardo Padilla Torres for their generous assistance.

The Ixtapa Conference attracted 120 participants from 23 countries: 27 from México, 25 from the United States, 10 each from the United Kingdom and Poland, 9 from Germany, 5 from Spain, 4 each from Japan and Russia, 3 each from Canada, Italy, Portugal, and Sweeden, 2 each from Belgium, Brazil, and China, and single representives from Austria, Cuba, Egypt, Finland, Israel, The Netherlands, Slovenia, and Switzerland. During the 6 days, there were 119 lectures, mostly in two parallel sessions.

2. THE SEVEN TOPICAL SESSIONS

Dedication to Gian-Carlo Rota. Gian-Carlo Rota and Joel A. Stein in 1994 raised the problem of accommodating Clifford algebra with the Hopf gebra or bigebra structure (this is the Bourbaki terminology instead of bialgebra, etc (Bourbaki, 1989)) and concluded that no such structure is possible (Rota and Stein, 1994, p. 13058). I considered the Chevalley deformation of Woronowicz's braided generalization of an exterior algebra (and cogebra) leading to a braided Clifford algebra. After Micho Đurđevich arrived in Mexico in 1993, we began stimulating discussions on this subject, which resulted in several separate and joint publications: on Clifford algebra for a Hecke braid (Oziewicz, 1995), on Clifford algebra for arbitrary braid (Đurđevich and Oziewicz, 1994, 1996), and on Clifford quantum (i.e., Hopf) algebra (Đurđevich, 1994, 2001). However, at this time I started to believe that the notion which must be studied first is Clifford cogebra alone, and that the question of how to define the Clifford bigebra in terms of a joint pair algebra and cogebra can be studied subsequently (Oziewicz, 1997, 1998) in the spirit of a Lie bigebra (cf. Michaelis, 1980). In meantime, Đurđevich defined a Clifford Hopf gebra without Clifford cogebra (1994, 2000).

Starting in 1995, I had many inspiring electronic discussions with Gian-Carlo Rota concerning how to accommodate the Hopf gebra or bigebra structure within a Clifford algebra. In 1997, we meet at the AMS Meeting in Oaxaca, Mexico, where we also arranged for him to give the main invited plenary lecture at the Fifth Conference on Clifford Algebra in Ixtapa, Mexico. Gian-Carlo Rota died at the end of April 1999, just before the Conference. I arranged a special session for the first day of the Conference dedicated to the memory of Prof. Rota with five lectures by scientists who knew him personally, **David Ritz Finkelstein, Bernd Schmeikal, Zbigniew Oziewicz,** and **Luis Verde-Star**. **Leopoldo Román Cuevas** was forced to withdraw at the last moment for health problems.

ACACSE'99. Eduardo Bayro-Corrochano and Garret Sobczyk organized a 2-day Workshop on Applied Clifford Algebra in Computer Science (Cybernetics, Robotics, Computer Vision, Neural Computing, Image Processing) and Engineering. This led to a published volume edited by Bayro-Corrochano and Sobczyk (2001) that consists of 25 contributions.

Clifford Analysis. This session on Local and Global Problems in Clifford Analysis took place over the 2 days of the Ixtapa Conference and was organized by Enrique Ramirez de Arellano, John Ryan, and Wolfgang Sprößig.

Spinor Structures and Dirac Operators. This session was organized by Micho Đurđevich.

Nonassociative Structures. Lev Sabinin organized the session on Moufang quasigroups and loops, which can be understand also as a triple of binary operations related by identities, generalizing a group operation to nonassociative structure.

Gravity and Elementary Particles. This session was organized by Leopold Halpern and Kurt Just.

Mathematical Physics. This session was organized by David Ritz Finkelstein.

The authors had several options for the submission of their papers: either to the volumes edited by Rafał Abłamowicz and Bertfried Fauser (2000), John Ryan and Wolfgang Sprößig (2000), and Eduardo Bayro-Corrochano and Garret Sobczyk (2001), or to the present issue of the *International Journal of Theoretical Physics*. Some authors used this possibility to increase their number of publications, because scholars are evaluated based on quantity rather than quality. Thus, some papers were artificially divided up for double or triple submission. Editors and referees heroically cooperated in order to avoid overlapings and duplications. Therefore sometimes it is necessary to read all the papers written by the same author(s) and scattered among different publications, and we sincerely suggest the reader do this.

David Hestenes and many of his followers are convinced that *"the much larger community"* of scientists and engineers is interested in the powerful tools of geometric algebra, as opposite to *Clifford* algebra. However, besides these irrelevant different names, there is also a deeper diversity of the philosophies that cannot be discussed so easily (Hestenes, 1992).

I want to thank the authors for their contributions to the present volume and to the reviewers for their generous help. Most of these publications were reviewed by up to three independent referees. After referee reports, most of the contributions were expanded and several times revised.

We have tried to organize the present issue thematically, by grouping related contributions. This has not been completely successful as several authors span several areas.

2. DIRAC OPERATOR: CROSS RELATIONS

In the present volume most of the papers deal with the Dirac operators: from the minimal coupling, $\gamma \circ \nabla \sim \gamma^{\mu}(\partial_{\mu} - iA_{\mu})$, to the maximal coupling

(Kähler, 1960, 1961, 1962), $\gamma \circ (\nabla + c \otimes -)$ with c in the Clifford algebra of the differential multiforms $\mathcal{C}\ell(V, -) \sim V^{\wedge}$, see the last Section for more details,

$$c = \text{Scalar} + \text{Vector} + \text{Bivector} + \text{Trivector} + \cdots \in \mathcal{C}\ell.$$

If $B = \frac{1}{2} B_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu} \in V^{\wedge 2}$ then $\gamma \circ B = \frac{1}{2} \gamma^{\mu\nu} B_{\mu\nu}$ is said to be the *Pauli term*. There seems to be a general trend to generalize the Dirac equation to maximal coupling. In particular, this is strongly advocated in this volume by **Rolf Dahm** through a phenomenology of the strong interactions, the model of the proton, etc, and also by **José Gabriel Vargas** and **Douglas Graham Torr**. **Vargas** and **Torr's** paper in this volume must be read together with the paper, Clifford-valued Clifforms: geometric language for Dirac equation, in Abłamowicz and Fauser (2000). A Dirac operator with the maximal coupling, in particular with the Pauli coupling, is referred to as "the generalized Dirac operator" by **Jürgen Tolksdorf**. **Tolksdorf**'s paper also must be read together with another paper of his, Geometry of generalized Dirac operators and the Standard Model of particle physics, in Ryan and Spröߑig (2000).

One should expect that all other papers dealing here with the minimal coupling will be extended soon to maximal coupling $\nabla \to \nabla + c \otimes -$, or $\gamma^{\mu}\partial_{\mu} - i\gamma^{\mu}A_{\mu} + \frac{1}{2}\gamma B_{\mu\nu} + \ldots$. In particular, it would be very interesting to see the analysis made by **Eduardo Piña** extended to maximal coupling, including in particular the Pauli term. The same expectation holds for analysis of the Hestenes equation made here by **Bertfried Fauser**. One can also expect that the spectral problems considered by **Micho Đurđevich** and by **Robert Owczarek** for the minimal coupling can be extended to the maximal coupling (this seems to be not so easy).

Trautman *et al.*, in Dietrich *et al.* (1998), study the spectral problem for the *modified* Dirac operator. This modification involve an orientation (a chirality) γ_5 and it can be included as a particular case in the maximal coupling problem.

Kurt Just *et al.*, in the context of the quantized Dirac field, prove that a nonquantized maximal coupling Bose field is a functional of the Dirac field. It would be interesting to try to reiterate the same analysis in the framework of noncommutative algebra without Minkowski space but with an additional structure of the Clifford $\mathbb{C}$-algebra of a hermitian form. **Kurt Just** and **James Thevenot**, in the context of the quantized fields, raise the problem of the Pauli term, and their paper, Pauli term must be absent in the Dirac equation, is published in Abłamowicz and Fauser (2000).

Marcos Rosenbaum gives a fairly comprehensive overview of Alain Connes' noncommutative theory. However, an even more interesting subject, the Hopf algebra of Feynman diagrams, is reviewed in another paper by

Marcos Rosenbaum jointly with **J. David Vergara**, Dirac operator, Hopf algebra of renormalization and structure of space-time, published in Ablamowicz and Fauser (2000).

Eckehard W. Mielke, Leopold Halpern, and **Norma Susana Mankoč Borštnik** study the Dirac equation in the context of elementary particles and gravity.

It would be interesting also to consider the Clifford algebra and Dirac operator on the lifted algebra in the spirit of Yano and Ishihara (1973) and Kainz and Michor (1987). In particular, if T denotes the tangent lift, then the Clifford algebra $\mathcal{C}\ell(V, \eta)$ can be lifted to $\mathcal{C}\ell(TV, T\eta)$. One can expect the Dirac operator for a lifted metric $T\eta$ to be relevant to gravity.

3. POLEMIC GUIDE: SIGNATURE CHANGE, QUASIGROUPs, PSEUDOTWISTOR

David Miralles, Josep Manel Parra, and **Jayme Vaz,** Jr., deal with the change of the signature. Franco Israel Piazzese is concerned with a similar problem and his paper, Pythagorean metric in relativity, is published in Abłamowicz and Fauser (2000). What does 'the change of signature' mean? Piazzese's map (cf., Miralles *et al.*) depends on (1, 3)-splitting. Let $p \equiv (E, \mathbf{p}) \in V$ be a momentum in Minkowski $\mathbb{R}$-space with a scalar product $p^2 = E^2 - \mathbf{p}^2$. The Piazzese map is

$$p = (E, \mathbf{p}) \mapsto \left(\frac{p^2}{E}, \mathbf{p}\sqrt{\frac{p^2}{E^2}}\right)$$

Then Minkowski norm of p is the same as the Pythagorean norm of the image,

$$p^2 = E^2 - \mathbf{p}^2 = \left(\frac{p^2}{E}\right)^2 + \mathbf{p}^2 \frac{p^2}{E^2}$$

However, it seems that Piazzese's 'quasi-classical dynamics' as presented in Ixtapa has nothing to do with the above 'change of signature' map and is just a consequence of the identity for the Lorentz relativistic factor γ,

$$\gamma^{-2} \equiv 1 - \frac{\mathbf{u}^2}{c^2} \Rightarrow c^2 = \mathbf{u}^2 + c^2 \cdot \gamma^{-2}$$

Miralles *et al.* explore the fact that the sum of two diagonal metrics with different signatures, say the Minkowski g_{M} and Euclidean g_{E} metrics, is a degenerate metric $\frac{1}{2}(g_{\mathrm{M}} + g_{\mathrm{E}})$ and such degenerate metric can be constructed from a splitting by selecting some unit vector(s).

Nonassociative algebraic structures are considered in several papers: by **Artibano Micali**, by **Jerzy Kociński**, by **Lev Sabinin**, and also by **Larissa**

V. Sbitneva and by **Alexander I. Nesterov**. Contribution by Micali will be published in one of the next issues for editorial reasons, see also his contribution to the volume edited by Dietrich *et al.* (1998). Micali deals with an (associative) Clifford $\mathbb{k}$-algebra $\mathcal{C}\ell(V, \eta \in V^* \otimes V^*)$ for not necessarily associative nor necessarily unital $\mathbb{k}$-algebra (V, m) with a nontrivial weight $\omega \in \mathbf{alg}(V, \mathbb{k})$. A scalar product η depends on this given weight ω in rather complicated way.

Lev Sabinin has long time been reformulating a Riemannian differential geometry in terms of the smooth quasigroups and loops (= unital quasigroups), introduced by Ruth Moufang around 1935, instead of the Lie groups. A quasigroup is a nonassociative generalization of a group and can be understand, for example, as a triple of binary operations related by identities. This structure can be treated in the framework of Birkhoff's equational universal algebra and it would be desirable to study different axiomatics in the same way as in the group theory by trying to determine the minimal set of relations, etc., as in the program presented by Tarski (1968). It also would be interesting to study the extension theory of quasigroups, i.e., short exact sequences of quasigroups, in the spirit of the Eilenberg (1948) program, in a similar way to the extension of groups. One can expect that extensions of quasigroups should lead to the general theory of representations of quasigroups and in particular to the general theory of odules (extending the family of modules, bimodules, etc.). **Sbitnieva** demonstrates how naturally the $\mathbb{R}$-odule arises in special relativity. **Sabinin** introduces for the smooth loops ($\simeq$ Lie loops) the vector fields, Lie bracket (Lie algebra?), and (affine) connection, but without mentioning the Leibniz condition. I was expecting to see the (analogy of) the Leibniz condition for a derivation. In the differential geometry of the Lie groups, a vector field is by definition a derivation, so one can ask how this definition can, or cannot, be adopted to the smooth loop case?

Andreas Bette, and **Julian Ławrynowicz** and **Osamu Suzuki**, deal with the twistors invented by Roger Penrose and also studied independently, among many other, by Jan Rzewuski in Wrocław since the 1970s (Kocik and Rzewuski, 1996). Bette's review paper on the twistor approach to the Dirac equation is published in Abłamowicz and Fauser (2000) and this introductory paper should be read first. Another approach to the Dirac operator in the framework of the twistor bundle was presented by Gusiew-Czudżak and Keller (1997).

Ławrynowicz and Suzuki decided to submit two papers on the same subject, first published in Abłamowicz and Fauser (2000) with a continuation is published in the present issue. They generalize the Penrose program for some (not any) other signatures, introducing pseudotwistors and bitwistors. However, the reader (of both papers) may find difficult to understand this terminology (among others) because all motivations were omitted and the relation with Penrose's twistors is not explained. Fortunately, the papers contain an extensive list of references.

The name *spinor* have been introduced by Élie Cartan in 1913. However, the Pauli σ matrices and the Dirac γ matrices with all defining Clifford algebra relations (without these names) were already published by I. Schur in 1911. We add to this issue the first English translation of this historical paper by I. Schur. In Section VI (Paragraph 21) of this paper, Schur deals with a finite Clifford group and with a spinor representation of the Clifford algebra in terms of the tensor product of essentially Pauli matrices [Schur (1911), formulas (50) and ff.].

4. CLIFFORD COGEBRA AND CLIFFORD CONVOLUTION

My lecture at the special session dedicated to the memory of Gian-Carlo Rota was devoted to, among other things, Clifford cogebra. I hope that the next Conference has a session on Clifford Cogebra and Applications.

In Ixtapa there were five lectures on related subjects: quantum groups by Woronowicz (not submitted), diagramatic approach to Hopf gebra by Souza (appearing in a later issue for the reason of the late submission), bigebra factorization by Drabant (not submitted), general cogebra (not necessarily Clifford) by Borowiec and Vázquez Coutino (this volume p. 67), and my own lecture on Clifford cogebra and Clifford convolution.

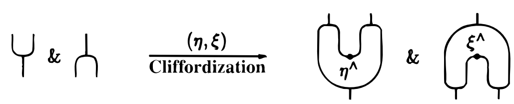
In this Section, I give a brief review of the motivation for the use of Clifford cogebra and Clifford convolution (Oziewicz, 1997, 1998; Cruz and Oziewicz, 2000; Fauser and Oziewicz, 2000). I believe that a Clifford cogebra must play an important role in all applications similar to a Clifford algebra.

In what follows, $\Bbbk$ is an associative and unital $\mathbb{N}$-algebra $\Bbbk \otimes_{\mathbb{N}} \Bbbk \to \Bbbk$ (a semiring), or a $\mathbb{Z}$-algebra $\Bbbk \otimes_{\mathbb{Z}} \Bbbk \to \Bbbk$ (a ring), not necessarily commutative, and $\otimes$ means $\otimes_{\Bbbk}$. Further, we will also need coscalars, i.e., $\mathbb{N}$-cogebra $\Bbbk \to \Bbbk \otimes_{\mathbb{N}} \Bbbk$ (a co-ring). A binary $\Bbbk$-algebra is a $\Bbbk$-bimodule V, an extension of $\Bbbk$, with a multiplication m_V as a $\Bbbk$-bimodule map $m_V\colon V \otimes V \to V$. This looks like a Feynman tree graph with one vertex m describing an annihilation. A binary $\Bbbk$-cogebra is like a creation process with a comultiplication $\Delta_V\colon V \to V \otimes V$. It is hard to believe that an algebra structure may be sufficient to explain all our problems in fundamental science, in applications, in engineering, in elementary particle physics, in logic, etc. The algebra and cogebra jointly give rise to convolution algebra and thus the name convolution. It was Heinz Hopf who discovered in 1941 that both these structures m and Δ, intertwine in algebraic topology, and thus the name Hopf gebra for an antipodal algebra and cogebra. The algebra and cogebra are like brother and sister, and it is unfortunate that present-day elementary textbooks on linear algebra do not mention this sister.

The reader not familar with semirings and bimodules can exchange a semiring for a field $\mathbb{R}$ and a $\mathbb{k}$-bimodule for a vector $\mathbb{R}$-space with almost no loss. The only difference is that in the case of a noncommutative $\mathbb{k}$, there are two different types of 'covectors', left and right covectors, because the right dual $\mathbb{k}$-bimodule V^* does not need to be the same as the left dual $\mathbb{k}$-bimodule $^*V$. In what follows, $(V \otimes V)^* \simeq V^* \otimes V^*$, etc.

If (V, Δ_V) is a model of a $\mathbb{k}$-cogebra Δ, then a dual $\mathbb{k}$-bimodule V^* with Δ_V^* is a $\mathbb{k}$-algebra (and also a left dual $\mathbb{k}$-bimodule $^*V$ with $^*\Delta_V$ is an another $\mathbb{k}$-algebra). In finite dimensions and in case of the graded dual in general, we have also the converse statement. In what follows, let V be a finite-dimensional $\mathbb{k}$-bimodule and a tensor $\eta \in V^* \otimes V^*$ be a $\mathbb{k}$-valued 'arbitrary bilinear form' on V, where $\mathbb{k}$ need not to be commutative. We do not need to assume that $\eta^T = \eta$, where η^T denotes transpose of η. Đurđevich's noncommutative algebra Σ in this issue is a particular $\mathbb{N}$-algebra or $\mathbb{Z}$-algebra or $\mathbb{R}$-algebra or $\mathbb{C}$-algebra and can be understand as our $\mathbb{k}$ in what follows.

Rota and Stein (1994) introduced a deformation of a convolution (of a Hopf gebra), called *Cliffordization*, a graphical 'sausage', which can be applied to a general convolution, not necessarily antipodal nor even unital. I showed in my lecture that Clifford cogebra alone can be obtained from an exterior Hopf gebra by analogous co-Cliffordization, dual to Rota and Stein's Cliffordization,



Here an exterior cogebra $\triangle\colon V^\wedge \to V^\wedge \otimes V^\wedge$ follows from the dual exterior algebra $\wedge\colon V^{*\wedge} \otimes V^{*\wedge} \to V^{*\wedge}$, $\triangle \equiv \wedge^*$, and can be calculated explicitly as follows, for $v, w \in V$:

$$\triangle 1 = 1 \otimes 1, \qquad \triangle v = 1 \otimes v + v \otimes 1,$$
$$\triangle(v \wedge w) = 1 \otimes (v \wedge w) - v \otimes w + w \otimes v + (v \wedge w) \otimes 1, \quad \text{etc.}$$

A tensor $\eta \in V^* \otimes V^*$ is a scalar product on V (coscalar on V^*), and $\xi \in V \otimes V$ is a coscalar product on V. These tensors lift to algebra maps $\eta^\wedge \in \mathbf{alg}(V^\wedge, V^{*\wedge})$ & $\xi^\wedge \in \mathbf{alg}(V^{*\wedge}, V^\wedge)$. Let $\mathrm{id}|V^{*\wedge n} \in V^{*\wedge n} \otimes V^{\wedge n}$ and $\mathrm{id} = \mathrm{id}_{V^{*\wedge}}$ or appropriately $\mathrm{id} = \mathrm{id}_{V^\wedge}$. Then for a basis $\{e_i \in V\}$ and a dual basis $\{\varepsilon^i \in V^*\}$, $\varepsilon^i e_j = \delta^i_j$, we have the braid-dependent expansions

$$\xi^\wedge = \xi^\wedge \circ \mathrm{id} = 1 \otimes 1 + \xi + \tfrac{1}{2}[(\xi\varepsilon^i) \wedge (\xi\varepsilon^j)] \otimes (e_j \wedge e_i) + \tfrac{1}{3!} \ldots \simeq \exp \xi$$

$$\eta^\wedge = \eta^\wedge \circ \mathrm{id} = 1 \otimes 1 + \eta + \tfrac{1}{2}[(\eta e_j) \wedge (\eta e_i)] \otimes (\varepsilon^i \wedge \varepsilon^j) + \tfrac{1}{3!} \ldots \simeq \exp \eta$$

An exponential of a tensor ξ, i.e., a lifted tensor $\xi^\wedge \simeq e^\xi$, in fact is braid dependent. A vertex $\xi^\wedge$ in a graphical sausage must be understand as a process $v \otimes w \mapsto v \otimes \xi^\wedge \otimes w$.

The Graßmann cogebra $(V^\wedge, \triangle)$ possess one group-like element only, namely 1, and therefore is a pointed irreducible cogebra. A Clifford cogebra possesses a discrete number of group-like elements and this number is correlated with the signature.

In the same spirit, one can treat *Wey*lization or *Heisenberg*ization of the symmetric exterior Weyl Hopf gebra. The quantization as the Moyal deformation of a symmetric multiplication (1949) in terms of the Poisson bivector field involves differential structure and most probably cannot be presented in as compact form as the above Rota and Stein Cliffordization.

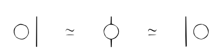
If $\mathcal{C}\ell(V, \eta) = (V^\wedge, \wedge^\eta)$, $\wedge^{\eta=0} \equiv \wedge$, is a Clifford $\mathbb{k}$-algebra ($\mathbb{R}$-algebra) as the η-Cliffordization of an exterior Hopf $\mathbb{k}$-algebra (or equivalently as the Chevalley deformation of an exterior $\mathbb{k}$-algebra) then $\mathcal{C}\ell(V^*, \eta) = (V^{*\wedge}, \wedge^{\eta^*})$ is a Clifford $\mathbb{k}$-cogebra ($\mathbb{R}$-cogebra). One can define also a universal Clifford $\mathbb{k}$-cogebra. In case a tensor η is invertible, we are dealing with a pair of mutually dual Clifford algebras of multivectors $\mathcal{C}\ell(V, \eta)$ and of multicovectors $\mathcal{C}\ell(V^*, \eta^{-1})$ as was explained elsewhere (Oziewicz, 1997, 1998). By duality, this gives that the $\mathbb{k}$-bimodules (or the $\mathbb{R}$-spaces) $V^\wedge$ & $V^{*\wedge}$ carry both structures, Clifford algebra and Clifford cogebra, and thus the name Clifford convolution.

Theorem 5.1 (Oziewicz 1997). The following unital and associative Clifford convolutions are antipode-less,

$$(V^\wedge, \wedge^\eta, \wedge^{\eta^{-1*}}) \quad \& \quad (V^{*\wedge}, \wedge^{\eta^{-1}}, \wedge^{\eta*}).$$

The above Theorem has been sharpened by Fauser and Oziewicz (2000): the Clifford convolution $\mathcal{C}\ell(\eta, \xi)$ is antipode-less iff $\det(\mathrm{id} - \xi \circ \eta) = 0$.

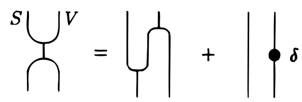
No attempt has been made yet to find axioms for the Clifford convolution $\mathcal{C}\ell(\eta, \eta^{-1})$. We believe that the set of such axioms may include the following



Indeed one can check that if $\eta^T = \eta$ then ${}^\wedge\eta \circ \Delta^{\eta^{-1}} = 2^{\dim V} \cdot \mathrm{id}_{\mathcal{C}\ell}$.

4.1. Coconnection and Dirac Operator for Clifford Cogebra

One can imagine the theory of coconnections as the coderivations. For this concept, we need a Leibniz coderivation from $\mathbb{k}$-bicomodule. To this end, let now $\mathbb{k}$ be $\mathbb{N}$-cogebra, V be $\mathbb{k}$-bicomodule (an extension of $\mathbb{k}$, *i.e.*, a short exact sequence $0 \to V \to V \oplus \mathbb{k} \to \mathbb{k} \to 0$), and let $\delta \in$ coder $(V, \mathbb{k})$ be a nontrivial Leibniz coderivation (Schlessinger and Stasheff, 1985; Borowiec and Vázquez Coutiño, 2000). In this case, we say that a $\mathbb{k}$-bicomodule V is a $\mathbb{k}$-bicomodule of the codifferential forms. An extension of an $\mathbb{N}$-cogebra $\mathbb{k}$ which is also an extension of a Clifford $\mathbb{k}$-cogebra $\mathcal{C}\ell(V, \eta^{-1})$ is said to be a $\mathbb{k}$-bicomodule of cospinors with a Clifford coaction $\gamma^*\ S \to S \otimes V$. Then a $\mathbb{k}$-comodule coderivation of cospinors S (left V-covariant), denoted by ∇^*: $\in$ coder$(S \otimes_{\mathbb{N}} V, S)$, defined by the following Leibniz condition, like Diagram 7.2 in (Oziewicz, 1998), is said to be a (spinor) δ-coconection:



For those who do not like operad of graphs, the above Leibniz condition can be translated as: $s \circ \nabla^* = (\nabla^* \otimes_{\mathbb{N}} \mathrm{id}) \circ (\mathrm{id} \otimes_{\mathbb{N}} v) + \mathrm{id} \otimes_{\mathbb{k}} \delta$.

A covariant coderivation ∇^* composed with Clifford coaction γ^* (for Clifford comodules) gives Dirac operator $\nabla^* \circ \gamma^*$ for Clifford cogebra. For all such concepts, we need first of all a notion of a Clifford cogebra. The proper, algebraic understanding of the spinor connection needs both Clifford module and Clifford comodule (Oziewicz, 1998; Gusiew-Czudżak and Oziewicz, 2000).

To talk about cotorsion and cocurvature for a coderivation ∇^*, we first need a De Rham cocomplex (noncommutative, because $\mathbb{k}$ needs to be neither commutative nor cocommutative),

$$\ldots \xrightarrow{\delta} V^{\wedge 2} \xrightarrow{\delta} V \xrightarrow{\delta} \mathbb{k}$$

A covariant coderivation ∇^* has a unique lift to cominimal coupling.

Finally, I hope to see in the near future a Clifford coanalysis as an extension of (co?)-holomorphic coderivations for (co)-complex (co?)-field with Cauchy–Riemann operator for Clifford cogebra. Why not Clifford bianalysis with joint derivations and coderivations?

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